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Fock space representations for non-Hermitian Hamiltonians

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Abstract

The requirement of Hermiticity of a quantum mechanical Hamiltonian, for the description of physical processes with real eigenvalues which has been challenged notably by Carl Bender, is examined for the case of a Fock space Hamiltonian which is bilinear in two creation and annihilation operators. An interpretation of this model as a Schrödinger operator leads to an identification of the Hermitian form of the Hamiltonian as the Landau model of a charged particle in a plane, interacting with a constant magnetic field at right angles to the plane. When the parameters of the Hamiltonian are suitably adjusted to make it non-Hermitian, the model represents two harmonic oscillators at right angles interacting with a constant magnetic field in the third direction, but with a pure imaginary coupling, and real energy eigenvalues. It is now $\mathcal{PT}$ symmetric. Multiparticle states are investigated.

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1. Introduction

The familiar Hamiltonians of quantum mechanics may be analysed for symmetries either in terms of pure matrix algebra, or else in terms of a Fock space representation, which generally leads to a more physical interpretation of the mathematical manipulations. In particular, any Hamiltonian constructed from a Fock space of n fermionic creation and annihilation operators may be transcribed in terms of a finite-dimensional matrix of dimension $2^{\frac{n}{2}} \times 2^{\frac{n}{2}}$ because such operators can always be constructed from gamma matrices, which admit a well-known matrix representation. The inverse of this construction, to obtain a Fock space representation of any finite-dimensional Hermitian matrix, can be done by embedding the matrix in a larger one of suitable dimension to admit representations of the canonical anti-commutation relations.

In fact, in view of the existence of a matrix representation for Hamiltonians the transcription is merely an exercise in matrix algebra for undergraduates. However, the situation is more complicated when the Hamiltonian is no longer Hermitian. It is a central tenet of quantum theory and quantum field theory that the Hamiltonian should be Hermitian, but this is not in fact necessary to maintain real eigenvalues, as has been demonstrated in recent years, notably by Bender and Boettcher [2]. However, if the eigenvalues are all discrete and real and their eigenvectors span the full space, there will exist, in general, an infinity of Hermitian Hamiltonians (possibly infinite dimensional) which are unitarily equivalent to the diagonal (real) form. They are equivalent up to a more general (not necessarily unitary) transformation to the initial Hamiltonian.

If all the eigenvalues are real and their eigenvectors span a subspace S of the initial space (say finite dimensional to make things simpler) there exists, in general, an infinity of Hamiltonians unitarily equivalent to a Jordan form (or a generalized Jordan form in the infinite-dimensional case) and equivalent up to a more general (non-necessary unitary) to the initial Hamiltonian. These Hamiltonians projected to the subspace S are Hermitian.

If some of the eigenvalues are discrete and real, this same statement will obviously apply in the subspace of the corresponding eigenvectors. We shall study a simple example which illustrates this in a physical context, that of the Landau problem of a particle in two dimensions moving under the influence of a constant magnetic field in the third direction [5, 6], and show that this model exhibits the features which have been found for $\mathcal{PT}$ symmetric non-Hermitian systems. In particular, while the usual model is Hermitian, with a real magnetic coupling, a deformation of the parameters in this model leads under certain systems to a non-Hermitian system with real eigenvalues and an imaginary coupling. In this respect it recalls to mind the work of Hollowood [4], who showed that affine Toda field theory with pure imaginary coupling possesses real energy levels. This system represents an anisotropic oscillator in interaction with a constant magnetic field. Some discussion is given of the case of transition between the Hermitian and non-Hermitian cases, where two of the eigenvalues coincide, and the associated matrix cannot be fully diagonalized, but can be reduced by similarity transformations only to Jordan normal form.

2. Two-dimensional case

Take a 2×2 matrix

$$M = \begin{bmatrix} s_1 & s_3 \\ s_4 & s_2 \end{bmatrix}. \qquad (1)$$

Then any finite-dimensional matrix eigenvalue problem can always be transformed to an infinite-dimensional quantum mechanical eigenvalue problem by the introduction of Fock space creation and annihilation operators $a_i^\dagger, a_j$, so that matrix M_{ij} turns into the Hamiltonian $\sum_{i,j} a_i^\dagger M_{ij} a_j$. This gives in this case the Hamiltonian

$$H = s_1 a_1^\dagger a_1 + s_2 a_2^\dagger a_2 + s_3 a_1^\dagger a_2 + s_4 a_2^\dagger a_1, \qquad (2)$$

where the coefficients $s_i, i = 1, \ldots, 4$ are complex constants, and $a_i^\dagger, a_i$ are two creation and annihilation operators satisfying the usual canonical commutation relations, with only the following nonzero commutators:

$$a_i a_j^\dagger - a_j^\dagger a_i = \delta_{ij}. \qquad (3)$$

The vacuum state $|0\rangle$ is defined as usual as the normalized state which is annihilated by the a

$$a_i|0\rangle = \mathcal{O} \qquad (4)$$

where $\mathcal{O}$ is the zero norm state in the Hilbert space.

3. Diagonalization

Take Hamiltonian (2) in the generic case for arbitrary complex values of s_i. Define

$$\Delta = \sqrt{(s_1 - s_2)^2 + 4s_3s_4}; \qquad \lambda_{\pm} = \tfrac{1}{2}(s_2 - s_1) \pm \tfrac{1}{2}\Delta, \tag{5}$$

and construct the linear combinations

$$\alpha_1^+ = n_1 \left(\frac{s_3}{\lambda_+} a_1^\dagger + a_2^\dagger \right) \tag{6}$$

$$\alpha_2^+ = n_2 \left(\frac{s_3}{\lambda_-} a_1^\dagger + a_2^\dagger \right) \tag{7}$$

$$\alpha_1 = \frac{1}{n_1} \left(\frac{s_4}{\Delta} a_1 + \frac{\lambda_+}{\Delta} a_2 \right) \tag{8}$$

$$\alpha_2 = \frac{1}{n_2} \left(\frac{-s_4}{\Delta} a_1 + \frac{-\lambda_-}{\Delta} a_2 \right) \tag{9}$$

where n_1 and n_2 are arbitrary factors. In this construction, α_j^+ denotes a creation operator, which should not be thought of as the Hermitian conjugate of α_j which is also an annihilation operator, since in the generic case (more precisely if $s_3/s_4^* \neq (s_2 - s_1)/(s_2 - s_1)^*$) one finds that $\alpha_j^\dagger$ cannot be made equal to α_j^+. However the properties

$$\langle 0|\alpha_i \alpha_j^+|0\rangle = \delta_{ij} \tag{10}$$

hold. Indeed the α_j and α_j^+ satisfy exactly the same commutation relations as do the a, because matrix (1) can be diagonalized by a similarity transform.

This implies that observables in the theory can be calculated, as n-particle states will have the form

$$|r, n - r\rangle = \frac{1}{\sqrt{r!(n-r)!}} \left(\alpha_1^+\right)^r \left(\alpha_2^+\right)^{n-r} |0\rangle. \tag{11}$$

These states are orthogonal to the following adjoint states:

$$|\mathrm{adj}[r, n - r]\rangle = \frac{1}{\sqrt{r!(n-r)!}} \left(\alpha_1^\dagger\right)^r \left(\alpha_2^\dagger\right)^{n-r} |0\rangle \tag{12}$$

in the sense that

$$\langle \mathrm{adj}[p', q']|p, q\rangle = \delta_{p'p}\delta_{q'q}. \tag{13}$$

The limitation of the use of non-Hermitian Hamiltonians is that the notion of + conjugation and reality of eigenvalues is specific to the Hamiltonian used and is not universal as is the case with Hermitian conjugation, as Bender *et al* have remarked [1]. Hermiticity also guarantees reality of eigenvalues, independently of the details of the Hamiltonian. This universality is a consequence of the fact that the inverse of a unitary matrix, which diagonalizes a Hermitian matrix, is its own Hermitian conjugate, i.e. since $U^\dagger = U^{-1}$ for a unitary matrix U the columns of $U^\dagger$ are orthogonal to the rows of U, the operation of Hermitian conjugation works independently of the Hermitian matrix to be diagonalized. In the case of a 2×2 matrix H we have

$$\begin{aligned} H &= s_1 a_1^\dagger a_1 + s_2 a_2^\dagger a_2 + s_3 a_1^\dagger a_2 + s_4 a_2^\dagger a_1 \\ &= \tfrac{1}{2}(s_1 + s_2 + \Delta)\, \alpha_1^+ \alpha_1 + \tfrac{1}{2}(s_1 + s_2 - \Delta)\, \alpha_2^+ \alpha_2. \end{aligned} \tag{14}$$

This construction shows that the n-particle states have energies of the form

$$E_{n,m} = \frac{n}{2}(s_1 + s_2) + \frac{m}{2}\Delta, \tag{15}$$

where m runs from $-n$ to n in steps of 2. There are several interesting features of this result. The eigenvalues will be real and distinct provided $s_1 + s_2$ is real and $(s_1 - s_2)^2 + 4s_3s_4$ is real and positive, thus even when the Hamiltonian is non-Hermitian, the eigenvalues may be real. When this latter factor is zero, then the eigenvalues are degenerate. Then, either the Hamiltonian is proportional to the unit matrix ($s_1 = s_2$, $s_3 = s_4 = 0$) or, if at least one off-diagonal element is nonzero, there is one eigenvector only and the Hamiltonian can be brought to the normal form ($s_1' = s_2'$, $s_3' = 0$, $s_4' = 1$). These cases are briefly discussed later. For non-Hermitian Hamiltonians, Bender *et al* [1] have analysed the existence of a Parity operator $\mathcal{P}$ and a conjugation matrix $\mathcal{PT}$. These crucial issues will be taken up and generalized after we demonstrate a simple application to a physical example of the 2×2 situation detailed above.

4. Schrödinger interpretation

The above problem is equivalent up to a constant energy shift to solving a Schrödinger equation for a particular quantum mechanical problem; set $a_i = p_i + \mathrm{i}q_i$ etc; then the problem is equivalent to

$$\begin{aligned} H = -s_1\left(\frac{\partial^2}{\partial x^2} - x^2\right) - s_2\left(\frac{\partial^2}{\partial y^2} - y^2\right) \\ - (s_3 + s_4)\left(\frac{\partial^2}{\partial x \partial y} - xy\right) + (s_3 - s_4)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \end{aligned} \tag{16}$$

up to a c-number addition. If $s_3 = -s_4$, then this is the Landau problem of a particle in a plane coupled to a magnetic field in the perpendicular direction, through its angular momentum with possibly an additional linear central force. To see this consider a constant magnetic field with potential $\vec{A} = \frac{B}{2}(x)$ in the Hamiltonian

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 = \frac{1}{2m}\left(p_x^2 + p_y^2 + \left(\frac{eB}{2c}\right)^2(x^2 + y^2) + \frac{eB}{c}(p_y x - p_x y)\right). \tag{17}$$

and make the identifications; $s_1 = s_2 = 1/(2m)$, and $s_3 = -s_4$ is pure imaginary. This Hamiltonian is Hermitian with real eigenvalues. However, the choice of $s_3 = -s_4$ and real, eliminates the xy cross terms and also gives real eigenvalues, provided that the factor $(s_1 - s_2)^2 - 4s_4^2$ is positive which necessarily entails that $s_1 \neq s_2$. In this case the Hamiltonian is non-Hermitian and it represents a constant magnetic field coupled to an anisotropic oscillator with pure imaginary coupling. In all the examples Bender *et al* [2, 3] have constructed with non-Hermitian Hamiltonians and real eigenvalues the Hamiltonian is $\mathcal{PT}$ symmetric. Dorey, Dunning and Tateo [7] have recently given a proof that the spectra of a number of $\mathcal{PT}$ invariant Hamiltonians are entirely real. However, on its own $\mathcal{PT}$ invariance only implies real or complex conjugate pairs of eigenvalues. This is the situation under consideration here as $\mathcal{PT}$ invariance means invariance under $(x, y) \to (-x, -y)$ and $i \to -i$. At the same time $\vec{p}$ does not change. The Hamiltonian will then be $\mathcal{PT}$ symmetric since $B \to B$, but the eigenvalues may be complex conjugate if the positivity condition is violated.

5. Real eigenvalue conditions

As is well known, the eigenvalues of a Hermitian matrix are always real. In general the condition for the reality of eigenvalues depends on more specific details of the matrix. However in the case of an $n \times n$ matrix H necessary conditions for the existence of real eigenvalues are given in terms of powers of H by

$$\mathrm{Tr}(H^r) = \text{real}; \qquad r = 1, 2, \ldots, n. \tag{18}$$

This result follows simply from the observations that the characteristic polynomial of H, being an invariant under similarity transformations, has coefficients expressible in terms of the traces of powers of H and that if all the eigenvalues are real, $H^\dagger$ must have the same eigenvalues as H. A further necessary and sufficient condition for an arbitrary 2×2 matrix to possess real eigenvalues is given by the requirement that $2\mathrm{Tr}\, H^2 - (\mathrm{Tr}\, H)^2 \geqslant 0$. This expression is just a translation into invariant form of the condition of given in section 3 after (15). Unfortunately, an invariant criterion even in the three-dimensional case is rather complicated. When the traces are real and $\Delta = 0$, the two-dimensional matrix H may be expressed as

$$H = \tfrac{1}{2}(s_1 + s_2)\mathbb{1} + A \tag{19}$$

where A is null or the zero matrix; i.e. $\mathrm{Tr}\, A = \mathrm{Tr}\, A^2 = 0$ or $A = 0$. In the former case the matrix is equivalent by a change of basis to a Jordan normal form

$$H = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}. \tag{20}$$

6. 'Conjugation' and 'parity': finite-dimensional case

In this section, we extend the notions of 'conjugation', and/or of 'parity' introduced in [2] to arbitrary finite-dimensional matrices representing the Hamiltonian with real eigenvalues. In certain cases the arguments can be extended immediately to infinite-dimensional spaces (see section 8 for an example). It is also not difficult to extend them to the case where some of the eigenvalues are equal or are complex or when the starting matrix M_d below has parts in a Jordan form. These concepts are familiar in the case of Hermitian Hamiltonians, and are discussed here to make contact with the important property of $\mathcal{PT}$ symmetry discussed by Bender *et al*.

We first recall a few well-known facts. Let M_d be a real diagonal matrix in p dimensions (with p distinct real diagonal elements μ_i), and let N be an arbitrary invertible $p \times p$ complex matrix. Let the scalar product of the column eigenvectors ψ and ϕ be defined by the obvious

$$\langle \psi | \phi \rangle = \psi^\dagger \phi \tag{21}$$

where $A^\dagger = (A^*)^{\mathrm{tp}}$ is the Hermitian conjugate matrix.

For any matrix M which has the eigenvalues μ_i there exists a matrix N such that

$$M = N M_d N^{-1} \quad \Leftrightarrow \quad M_d = N^{-1} M N. \tag{22}$$

The p vectors ϕ_i, $i = 1, \ldots, p$, with components

$$(\phi_i)_j = \delta_{ij}, \tag{23}$$

are obviously eigenvectors of M_d with eigenvalues μ_i. They are orthonormal for the scalar product

$$\langle \phi_j | \phi_k \rangle = \delta_{jk}. \tag{24}$$

The vectors $\psi_i = N\phi_i$ are obviously eigenvectors of the general M with the same eigenvalue.

We now present a few results providing, in this rather general case, the general solution to the question of existence and construction of a suitable scalar product $\langle \phi \mid \psi \rangle_G$, of a 'parity' operator P and of a 'conjugation' operator C.

(1) The vectors ψ_i are orthonormal for the Hermitian scalar product defined by

$$\langle \psi | \phi \rangle_{G} = \psi^{\dagger} G \phi, \qquad \text{with} \quad G = (N^{-1})^{\dagger} N^{-1}. \tag{25}$$

Indeed

$$\begin{aligned}\delta_{jk} &= \langle \phi_j | \phi_k \rangle \\ &= \langle N^{-1}\psi_j | N^{-1}\psi_k \rangle \\ &= \psi_j^{\dagger} (N^{-1})^{\dagger} N^{-1} \psi_k \\ &= \psi_j^{\dagger} G \psi_k \\ &= \langle \psi_j | \psi_k \rangle_{G}.\end{aligned} \tag{26}$$

Because of the form of G (25) all the requirements of the scalar product are met. Note however that if N is unitary, then G is the identity matrix; otherwise, it depends upon the structure of N. This means that while Hermitian conjugation can be defined universally for Hermitian Hamiltonians, in the non-Hermitian case, it is a more specific matrix. Two matrices M lead to the same metric G if the N which defines the first one is equal to the N which defines the second multiplied by an arbitrary unitary matrix.

(2) We denote complex by $\mathcal{T}$ the symbolic operator which performs the c-number conjugation, $\mathcal{T}$

$$\mathcal{T} A = A^* \mathcal{T}. \tag{27}$$

In the following we now prove that there exists always a matrix P (generically called 'parity' as Bender suggested) which has the following property:

$$(P\mathcal{T})M = M(P\mathcal{T}) \qquad PM^* = MP. \tag{28}$$

Take any invertible matrix K which commutes with M_d,

$$KM_d = M_d K. \tag{29}$$

In particular any diagonal matrix will do if the eigenvalues since are all different. If some eigenvalues are equal, K may belong to the stability group of M_d. Then

$$P = NK(N^*)^{-1}. \tag{30}$$

Indeed, from (28), we find

$$\begin{aligned}&P(NM_dN^{-1})^* = NM_dN^{-1}P \quad \text{by (22)} \\ &PN^*M_d(N^{-1})^* = NM_dN^{-1}P \\ &(N^{-1}PN^*)M_d = M_d(N^{-1}PN^*).\end{aligned} \tag{31}$$

Using (30) and (31), the statement is proved.

(3) The matrix $\mathcal{C}$ (conjugation) is defined as having the following property:

$$CM = MC \qquad C^2 = 1. \tag{32}$$

The general solution for C is

$$C = NK_sN^{-1} \tag{33}$$

where K_s is such it commutes with M_d and $K_s^2 = \mathbb{1}$. In the generic case, K_s is a diagonal matrix with elements $\pm$ only. Up to a trivial overall sign, there are 2^{p-1} independent C. Indeed the first equation of (28) gives

$$\begin{aligned} &CM = MC \qquad CNM_dN^{-1} = NM_dN^{-1}C \quad \text{by (22)} \\ &(N^{-1}CN)M_d = M_d(N^{-1}CN) \end{aligned} \tag{34}$$

which implies that $N^{-1}CN = K_s$ commutes with M_d. The second equation implies moreover that

$$K_s^2 = \mathbb{1}. \tag{35}$$

In the two-dimensional example given in section 3 after (15), C is either the identity matrix or the matrix

$$C = \frac{1}{\Delta}\begin{bmatrix} s_1 - s_2 & 2s_3 \\ 2s_4 & s_2 - s_1 \end{bmatrix} \tag{36}$$

up to sign.

7. More bosonic creation operators

In this section, we elaborate on two- or more-particle states obtained for the simplest Hamiltonian (2). We will treat in great detail the two- and four-particle states.

7.1. Two-particle states

Let us first elaborate on the two-particle states constructed from the operators of section 2. Suppose that we were to start with the products $b_1^\dagger$, $i = 1, 2, 3$ of two creation operators

$$b_1^\dagger = \left(a_1^\dagger\right)^2 \qquad b_2^\dagger = a_1^\dagger a_2^\dagger \qquad b_3^\dagger = \left(a_2^\dagger\right)^2 \tag{37}$$

which construct the three two-particle states $b_i^\dagger|0\rangle$, $i = 1, 2, 3$ when applied to the vacuum of the a_i operators. The corresponding three-dimensional matrix M is

$$M^{[2]} = \begin{pmatrix} 2s_1 & s_3 & 0 \\ 2s_4 & s_1 + s_2 & 2s_3 \\ 0 & s_4 & 2s_2 \end{pmatrix}. \tag{38}$$

It satisfies (with H from (2))

$$\left[H, b_i^\dagger\right] = M_{ji}^{[2]} b_j^\dagger \tag{39}$$

and hence

$$Hb_i^\dagger|0\rangle = M_{ji}^{[2]} b_j^\dagger|0\rangle. \tag{40}$$

The diagonalization of M provides the two-particle spectrum. Using Δ from (5), we find, as expected, three eigenstates β_i^+ which fulfil the eigenvalue equation (with H from (2))

$$H\beta_i^+|0\rangle = \mu_i \beta_i^+|0\rangle \tag{41}$$

They are up to a factor, obviously from (2)

$$\begin{aligned} \beta_1^+ &= -2s_3^2 b_1^\dagger + 2s_3(-\Delta + s_1 - s_2)b_2^\dagger + (\Delta(s_1 - s_2) - (s_1 - s_2)^2 - 2s_3s_4)b_3^\dagger \propto \left(\alpha_1^+\right)^2 \\ \beta_2^+ &= -s_3 b_1^\dagger + (s_1 + s_2)b_2^\dagger + s_4 b_3^\dagger \propto \alpha_1^+\alpha_2^+ \\ \beta_3^+ &= -2s_3^2 b_1^\dagger + 2s_3(\Delta + s_1 - s_2)b_2^\dagger + (-\Delta(s_1 - s_2) - (s_1 - s_2)^2 - 2s_3s_4)b_3^\dagger \propto \left(\alpha_2^+\right)^2 \end{aligned} \tag{42}$$

with eigenvalues (see (15))

$$\begin{aligned}\mu_1 &= s_1 + s_2 + \Delta = E_{2,2} = 2E_{1,1}\\ \mu_2 &= s_1 + s_2 = E_{2,0} = E_{1,1} + E_{1,-1}\\ \mu_3 &= s_1 + s_2 - \Delta = E_{2,-2} = 2E_{1,-1}.\end{aligned} \tag{43}$$

This discussion suggests the following question: 'how can this spectrum be obtained directly from the operators defining the two-particle space?' The underlying problem with the construction of a two-particle Hamiltonian out of the single-particle operators is that it is impossible to build operators which we call $\widetilde{b}_i$ satisfying canonical commutation relations. To remedy this difficulty, two paths can be tried.

(1) A first naïve attempt is as follows. Introduce symbolically the new operators $\widetilde{b}_i$ which are supposed, with the $b_j^\dagger$, to satisfy the canonical commutation relations

$$\left[\widetilde{b}_i, b_j^\dagger\right] = \delta_{ij} \tag{44}$$

and define a two-particle Hamiltonian $H^{[2]}_{\text{naive}}$ as

$$H^{[2]}_{\text{naive}} = b_i^\dagger M_{ij}^{[2]} \widetilde{b}_j. \tag{45}$$

Then obviously but, we insist, symbolically we obtain

$$\left[H^{[2]}_{\text{naive}}, b_i^\dagger\right] = M_{ji}^{[2]} b_j^\dagger \tag{46}$$

and on a vacuum $|0^{[2]}\rangle$ such that

$$\widetilde{b}_j |0^{[2]}\rangle = 0 \tag{47}$$

we obtain

$$H^{[2]}_{\text{naive}} b_i^\dagger |0^{[2]}\rangle = M_{ji}^{[2]} b_j^\dagger |0^{[2]}\rangle. \tag{48}$$

It is not difficult to see that it is impossible to construct $\widetilde{b}_i$ satisfying (44) out of products of two a_j. However if the vacuum $|0^{[2]}\rangle$ is thought to be the a vacuum $|0\rangle$

$$|0^{[2]}\rangle := |0\rangle \tag{49}$$

and the weaker condition (compared with (44))

$$\left[\widetilde{b}_i, b_j^\dagger\right]|0\rangle = \widetilde{b}_i b_j^\dagger |0\rangle = \delta_{ij}|0\rangle \tag{50}$$

is imposed, the $\widetilde{b}_i$ can be identified to be

$$\widetilde{b}_1 = \tfrac{1}{2} b_1, \qquad \widetilde{b}_2 = b_2, \qquad \widetilde{b}_3 = \tfrac{1}{2} b_3 \tag{51}$$

in the weak sense with, obviously, $b_1 = a_1^2, b_2 = a_1 a_2, b_3 = a_3^2$. Equation (48) then holds.

(2) In a second approach, one tries to construct directly an $H^{[2]}$ quadratic both in $a_i^\dagger$ and in a_i and which satisfies the basis equation (46)

$$\left[H^{[2]}, b_i^\dagger\right]|0\rangle = M_{ji}^{[2]} b_j^\dagger |0\rangle \tag{52}$$

as a result of the basic a_i canonical commutation relations (3).

Such a Hamiltonian does not exist in the strong sense (when the vacuum is removed in (52)). In the weak sense, it exists as

$$H^{[2]} = b_i^\dagger N_{ij}^{[2]} b_j, \tag{53}$$

where the matrix $N^{[2]}$ is

$$N^{[2]} = \begin{pmatrix} s_1 & s_3 & 0 \\ s_4 & s_1 + s_2 & s_3 \\ 0 & s_4 & s_2 \end{pmatrix}. \tag{54}$$

This result is obviously equivalent to the naïve approach once the $\widetilde{b}_i$ are expressed in terms of the b_i (51).

7.2. Four-particle states

Let us proceed in the same way for the four-particle states which can be thought either directly as the four-particle states in the original operators $\left(a_i^\dagger, i = 1, 2\right)$ (3)

$$d_i^\dagger = \left(a_1^\dagger\right)^{5-i}\left(a_2^\dagger\right)^{i-1}, \quad i = 1, \ldots, 5 \tag{55}$$

or as the compound states constructed out of two of the basic two-particle states $b_i^\dagger, i = 1, 2, 3$ (37)

$$\begin{aligned} \check{d}_1^\dagger &= \left(b_1^\dagger\right)^2, & \check{d}_2^\dagger &= b_1^\dagger b_2^\dagger, & \check{d}_3^\dagger &= b_1^\dagger b_3^\dagger, \\ \check{d}_4^\dagger &= \left(b_2^\dagger\right)^2, & \check{d}_5^\dagger &= b_2^\dagger b_3^\dagger, & \check{d}_6^\dagger &= \left(b_3^\dagger\right)^2. \end{aligned} \tag{56}$$

Let us note immediately that there are only five states built with the five $d_i^\dagger$ while there are six states built with the six $\check{d}_i^\dagger$. This is because

$$\check{d}_3^\dagger \equiv \check{d}_4^\dagger \tag{57}$$

when constructed from the a'.

Let us treat in succession the $d_i^\dagger$ case and the $\check{d}_i^\dagger$ case. In terms of the construction of (56), the states $\check{d}_3^\dagger|0^{[2]}\rangle$ and $\check{d}_4^\dagger|0^{[2]}\rangle$ both have the same energy. We are going to demonstrate the resolution of this situation by explicit calculation. Indeed, in general, starting from our original Hamiltonian (2), the four-particle state matrix, in terms of the $b_i^\dagger$, is not fully diagonalizable. It is only reducible to a Jordan normal form. This is one of the few instances where the occurrence of a Jordan form appears in a physical context.

- The 5×5 matrix $M^{[4]}$, analogous to (38) and defined by

$$\left[H, d_i^\dagger\right] = M_{ji}^{[4]} d_j^\dagger, \tag{58}$$

is

$$M^{[4]} = \begin{pmatrix} 4s_1 & s_3 & 0 & 0 & 0 \\ 4s_4 & 3s_1 + s_2 & 2s_3 & 0 & 0 \\ 0 & 3s_4 & 2(s_1 + s_2) & 3s_3 & 0 \\ 0 & 0 & 2s_4 & s_1 + 3s_2 & 4s_3 \\ 0 & 0 & 0 & s_4 & 4s_2 \end{pmatrix}. \tag{59}$$

From this expression, it is easy to generalize the general form for states with n original particles. Its eigenvalues and eigenstates, analogous to (42) and (43) can be read off explicitly in (11) and (15).

Note that, in the weak sense, there is a Hamiltonian $H^{[4]}$ which may be written directly in terms of the d_j and $d_j^\dagger$

$$\left[H^{[4]}, d_i^\dagger\right]|0\rangle = M_{ji}^{[4]} d_j^\dagger|0\rangle \tag{60}$$

and is

$$H^{[4]} = d_i^\dagger N_{ij}^{[4]} d_j \tag{61}$$

with

$$N^{[4]} = \frac{1}{6}\begin{pmatrix} s_1 & s_3 & 0 & 0 & 0 \\ s_4 & 3s_1 + s_2 & 3s_3 & 0 & 0 \\ 0 & 3s_4 & 3(s_1 + s_2) & 3s_3 & 0 \\ 0 & 0 & 3s_4 & s_1 + 3s_2 & s_3 \\ 0 & 0 & 0 & s_4 & s_2 \end{pmatrix}. \tag{62}$$

- Let us now try to define the 6×6 matrix $\check{M}^{[4]}$ in the same way for the $\check{d}_i^\dagger$

$$\left[H_{\text{naive}}^{[2]}, \check{d}_i^\dagger\right] = \check{M}_{ji}^{[4]} \check{d}_j^\dagger. \tag{63}$$

Again, starting from the same canonical commutation relations for the $b_i^\dagger$'s and $\widetilde{b}_j$ (44), we find

$$\check{M}_{ij}^{[4]} = \begin{bmatrix} 4s_1 & s_3 & 0 & 0 & 0 & 0 \\ 4s_4 & 3s_1 + s_2 & 2s_3 & 2s_3 & 0 & 0 \\ 0 & s_4 & 2(s_1+s_2) & 0 & s_3 & 0 \\ 0 & 2s_4 & 0 & 2(s_1+s_2) & 2s_3 & 0 \\ 0 & 0 & 2s_4 & 2s_4 & s_1 + 3s_2 & 4s_3 \\ 0 & 0 & 0 & 0 & s_4 & 4s_2 \end{bmatrix}. \tag{64}$$

Since the $d_i^\dagger$ are perfectly defined there is no ambiguity. However, coherence with (57) implies that an arbitrary combination of $\check{d}_3^\dagger - \check{d}_4^\dagger$ can be added to the right-hand side of (63), and $\check{M}_{ij}^{[4]'}$ can be replaced by

$$\check{M}_{ij}^{[4]'} = \check{M}_{ij}^{[4]} + r_j \delta_{i3} - r_j \delta_{i4}, \tag{65}$$

namely,

$$\check{M}_{ij}^{[4]'} = \begin{bmatrix} 4s_1 & s_3 & 0 & 0 & 0 & 0 \\ 4s_4 & 3s_1 + s_2 & 2s_3 & 2s_3 & 0 & 0 \\ r_1 & s_4 + r_2 & 2(s_1+s_2) + r_3 & r_4 & s_3 + r_5 & r_6 \\ -r_1 & 2s_4 - r_2 & -r_3 & 2(s_1+s_2) - r_4 & 2s_3 - r_5 & -r_6 \\ 0 & 0 & 2s_4 & 2s_4 & s_1 + 3s_2 & 4s_3 \\ 0 & 0 & 0 & 0 & s_4 & 4s_2 \end{bmatrix}. \tag{66}$$

Making the same combination for the left-hand side, it can easily be checked that

$$\left[H_{\text{naive}}^{[2]}, \check{d}_3^\dagger - \check{d}_4^\dagger\right] = (2s_1 + 2s_2 + r_3 - r_4)\left(\check{d}_3^\dagger - \check{d}_4^\dagger\right) \tag{67}$$

i.e. a combination of $\check{d}_3^\dagger - \check{d}_4^\dagger$ only. This result is coherent when $\check{d}_3^\dagger - \check{d}_4^\dagger$ is put to zero. The eigenvalues of the matrix $\check{M}_{ij}^{[4]'}$ depend upon the values of the arbitrary parameters r_i since the secular equation is

$$(\lambda - E_{4,4})(\lambda - E_{4,-4})(\lambda - E_{4,2})(\lambda - E_{4,-2})(\lambda - E_{4,0})(\lambda - E_{4,0} - r_3 + r_4). \tag{68}$$

When $r_4 \neq r_3$, the eigenvalues are all different and the matrix can be diagonalized. When $r_4 = r_3$, two eigenvalues become equal. Then there is always the eigenvector $(0, 0, 1, -1, 0, 0)^{\text{tp}}$. The condition for the existence of a second eigenvector is

$$r_6 s_4^2 + 2r_5 s_1 s_4 - 2r_5 s_2 s_4 + r_3 (s_1 - s_2)^2 - 2r_3 s_3 s_4 - 2r_2 s_1 s_3 + 2r_2 s_2 s_3 + r_1 s_3^2 = 0. \tag{69}$$

When this condition is satisfied, the matrix can be transformed to a fully diagonal form; otherwise it is reducible to a Jordan normal form

$$\begin{bmatrix} E_{4,4} & & 0 & 0 & 0 & 0 \\ 0 & E_{4,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & E_{4,0} & 1 & 0 & 0 \\ 0 & 0 & 0 & E_{4,0} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{4,-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{4,-4} \end{bmatrix}. \tag{70}$$

8. Extension to a Fock space

We shall now promote the discussion to the case of a Fock space Hamiltonian linear in the products of one creation and one annihilation operator $a_i^\dagger a_j$ of bosons of two (or more) different species. The operators N (analogous to the matrix N of (22)) now become functions of the creation and annihilation operators $\left(a_j, a_j^\dagger\right)$ which satisfy the usual canonical commutation relations (see (3)).

In analogy with the two-dimensional case (14) and with the discussion of the preceding section, the diagonal Hamiltonian is defined as

$$H_d = \sum_j \mu_j a_j^\dagger a_j \tag{71}$$

and the related Hamiltonian H by

$$H = N H_d N^{-1} \tag{72}$$

which should be linear in the products $a_i^\dagger a_j$ since this is a free field theory. Suppose

$$N a_i N^{-1} = \sum_j c_{ij}^{[1]} a_j \quad (\equiv \alpha_i) \qquad N a_i^\dagger N^{-1} = \sum_j d_{ij}^{[1]} a_j^\dagger \quad \left(\equiv \alpha_i^\dagger\right), \tag{73}$$

where the c_{ij} and d_{ij} are arbitrary complex numbers. The two related matrices are chosen to be invertible in order to keep the number of independent degrees of freedom unchanged. Indeed, we then find

$$H = \sum_j \mu_j \left(\sum_k d_{jk}^{[1]} a_k^\dagger\right)\left(\sum_m c_{jm}^{[1]} a_m\right) = \sum_k \sum_m \left(\sum_j \mu_j d_{jk}^{[1]} c_{jm}^{[1]}\right) a_k^\dagger a_m. \tag{74}$$

If an interpretation in terms of creation and annihilation operators is to remain, the right-hand sides of (73) should obey the same commutation relations (3) as the a. This implies for the matrices c and d having respectively c_{ij} and d_{ij} as components the restrictions

$$d^{[1]tp} = (c^{[1]})^{-1}. \tag{75}$$

After some algebra, using (75) and defining

$$d^{[2]} = d^{[1]} - \mathbb{1}, \tag{76}$$

one finds

$$[a_i, N] = d_{ij}^{[2]tp} N a_j \tag{77}$$

$$\left[N, a_i^\dagger\right] = d_{ij}^{[2]} a_j^\dagger N. \tag{78}$$

Using a power expansion of N in terms of the a, it is easy to show that the quadratic part of N is

$$Q = \sum_{ij} d_{ij}^{[2]} a_j^\dagger a_i, \tag{79}$$

and that the general N is constructed from Q as

$$N =: \exp(Q) : \tag{80}$$

where the symbol $: \ldots :$ denotes the normal product (annihilation operators written at the right of creation operators). Indeed, we have

$$\begin{aligned}[a_i, N] &= \partial_{a_i^\dagger} : \exp(Q) : \\ &=: \exp(Q) : \left(\partial_{a_i^\dagger} Q\right) \\ &= N\left(\sum_j d_{ji}^{[2]} a_j\right) \quad \Leftrightarrow \quad \text{equation (77)}\end{aligned} \tag{81}$$

but also

$$\begin{aligned}
\left[N, a_i^\dagger\right] &=: \partial_{a_i} \exp(Q) : \\
&=: \left(\partial_{a_i} Q\right) \exp(Q) : \\
&= \left(\sum_j d_{ij}^{[2]} a_j^\dagger\right) N \quad \Leftrightarrow \quad \text{equation (78).}
\end{aligned} \tag{82}$$

This completes the proof of (80).

It is then not difficult to see that, inversely, any H of the form

$$H = h_{ij} a_j^\dagger a_i \tag{83}$$

can be put into diagonal form H_d by using the inverse of N and adjusting the coefficient $c^{[1]}, d^{[1]}$ suitably. The μ are the eigenvalues of the matrix constructed with the h_{ij} which is supposed to be diagonalizable at this stage.

In particular for the case of $p = 2$, this can be read off directly for an arbitrary H from formulae (6)–(9) for the coefficient $c^{[1]}, d^{[1]}$ of equation (73).

Our arguments about the existence of a Hermitian Hamiltonian when the eigenvalues are real also applies to the infinite-dimensional case. While we were concluding this paper, a paper by Mostafadazeh [9] appeared, demonstrating that a quantum mechanical anharmonic oscillator with potential $\mathrm{i}x^3$ giving rise to a non-Hermitian Hamiltonian can be transformed to a Hermitian form, thus giving an alternative proof of the reality of energy eigenvalues.

9. Conclusions

The Landau problem of a particle in a magnetic field has been shown to demonstrate the phenomenon of a non-Hermitian Hamiltonian giving rise to real eigenvalues when the coupling to the magnetic field becomes pure imaginary. All properties of this model are explicitly calculable as it is really a free field theory in disguise, which illuminates the conditions under which non-Hermitian Hamiltonians may have real eigenvalues. Indeed, starting from real eigenvalues in a finite system, which is then subject to a similarity transform, it has been shown how to define creation and annihilation operators, and their conjugation to build a Fock space for a general non-Hermitian Hamiltonian. Since diagonalizable matrices with real eigenvalues are transformable by unitary transformations to Hermitian ones, the $\mathcal{PT}$ symmetric examples of Bender with $(\mathrm{i}x)^n$ potentials must be transformable to Hermitian systems. However our own attempt to find an equivalent Hermitian system led to an rather complicated one in terms of a power series, as did a similar attempt in the literature by Bender *et al*. Thus it may be that some systems exhibit their simplest form in a non-Hermitian, $\mathcal{PT}$ symmetric form.

Other properties of such systems have been discussed, notably the conditions for degenerate eigenvalues. In the 2×2 case and extensions thereof, this is a sign that the associated matrix is not fully diagonalizable and marks the transition between real and complex conjugate pairs of eigenvalues.

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References

[1] Bender C M, Brody D C and Jones H F 2003 Must a Hamiltonian be Hermitian? *Am. J. Phys.* **71** 1095–102 (*Preprint* hep-th/0303005)

[2] Bender C M and Boettcher S 1998 Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry *Phys. Rev. Lett.* **80** 4243–5246 (*Preprint* physics/9712001)

[3] Bender C M, Boettcher S and Meisinger P N 1999 $\mathcal{PT}$-symmetric quantum mechanics *J. Math. Phys.* **40** 2201–29 (*Preprint* hep-th/9809072)

[4] Hollowood T J 1992 Quantum solitons in affine toda field theories *Nucl. Phys.* B **384** 523 (*Preprint* hep-th/9110010)

[5] Gamboa J, Loewe M, Mendez F and Rojas J C 2001 The Landau problem and noncommutative quantum mechanics *Mod. Phys. Lett.* A **16** 2075–8 (*Preprint* hep-th/0104224)

[6] Dayi O F and Kelleyane L T 2002 Wigner functions for the Landau problem in noncommutative spaces *Mod. Phys. Lett.* A **17** 1937–44 (*Preprint* hep-th/0202062)

[7] Dorey P E, Dunning C and Tateo R 2001 Spectral equivalences, Bethe Ansatz equations, and reality properties in $\mathcal{PT}$-symmetric quantum mechanics *J. Phys. A: Math. Gen.* **34** 5679–703 (*Preprint* hep-th/0103051)

[8] Dorey P E, Dunning C and Tateo R 2001 Supersymmetry and the spontaneous breakdown of $\mathcal{PT}$ symmetry *J. Phys. A: Math. Gen.* **34** L391–400

[9] Mostafazadeh A 2004 $\mathcal{PT}$-Symmetric cubic anharmonic oscillator as a physical model *Preprint* quant-ph/0411137